

## Bayesian Inversion of Simply Modeled Magnetic Anomaly Data

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### ABSTRACT

The objective of this project was to develop a method to find a magnetic dipole buried beneath the earth's surface using Bayesian techniques to interpret magnetic field data collected at the surface. A computer was used to simulate experimentally collected noisy data. The so-called "forward problem" was used to help understand the Bayesian inverse problem solution technique to be used. From the experimental data generated, I was able to predict the location of a single dipole in two dimensions. The overall purpose of this project was to develop the beginnings of a quantitative technique that could be used to find magnetically active buried objects due to the earth magnetic field anomalies created. The simple case of a single magnetic dipole was investigated to indicate the potential problems that might occur with multiple dipoles in three dimensions.

Keywords: *Bayesian, magnetic field, and inverse problem*

### INTRODUCTION

The objective of my research was to find the spatial distribution, orientation, and strength of a simply magnetically interacting body buried beneath the earth's surface (a single dipole). I used Bayesian analysis of magnetic anomaly data measured above the earth's surface. I used a computer software program to simulate the data and then entered the data into a program to graph different vectors with different directions to tell me the orientation and strength.

Previous research has not taken into account dipole size and shape. It basically uses the magnetic field to form a contour map. Currently, work is on going on this problem at Pacific Northwest Laboratories. However, their work is on a broader scale. They are trying to find buried bunkers and hidden weapons (Cogbill 1995). I focused on the smaller problem.

I had to be able to effectively handle "noise" from the data for predictions to be accurate. The discrimination between the actual object and the noise around it is a very important step, due to the fact that if noise is interpreted as part of the object, skewed results will occur. I used standard magnetic field formulas to do the "forward" problem and then was able to use what I found to do the "inverse" problem.

Scientists are continuously attempting to solve a countless array of problems. These can generally be classified in two groups: (1) Direct problems, typically deterministic in nature, and (2) Inverse problems which are often ill-posed in the mathematical sense and, thus, usually require a probabilistic approach to find a solution. An example of an inverse problem from ordinary experience might be a physicians' diagnosis of a patients' illness. Given the symptoms (noisy, incomplete data) what is the disease (model), that is, given the effects what are the causes? Direct or forward problems usually start with a well-defined algorithm or determination process, which is then

used to produce noiseless, complete data. It is not hard to deduce that the majority of the problems faced in life, both in the technical sense and non-technical sense (e.g. daily decisions), belong in the inverse category.

One inverse problem recently investigated concerns the location of buried structures, which create anomalies in the earth's magnetic field in localized spatial regions (Lessor 1997). Specifically, man-made ferro-magnetic materials buried beneath the ground's surface will often create discernible earth magnetic field anomalies at the ground surface (Bhattacharya 1964, Talwani 1965, Traynin 1993). The question then becomes, "Can a suitable, quantitative approach be developed to analyze the earth's surface data, i.e., noisy magnetic field data, to answer a series of questions about what is buried below ground surface?" Questions addressed might be the location, size, shape, and orientation of a buried object as well as other questions tailored to the task at hand.

The approach of Lessor, et al, was to model any buried ferro-magnetic man-made structure as a complex of intermixed dipoles whose magnetic fields are excited by the earth's magnetic field and modified by self-interaction. The general buried object problem they posed is complex for many reasons, not the least of which being the presence of magnetic field noise resulting from naturally occurring anomalies or other man-made structures producing interference, but otherwise not of direct interest. Although they made progress on this problem, the simplest of situations was overlooked - a single buried dipole - to answer some basic questions, which remain open. Their questions address many different issues including range versus resolution, signal to noise degradation, coherent versus incoherent noise, and magnetic field magnitude data versus vector component data.

We focus here on a single buried magnetic dipole and how to characterize it using a direct probability

approach. This Bayesian approach, though not new, has yet to see wide spread use in inverse problem data analysis, although there is ample evidence of its power as a general data analysis tool (Bretthorst 1988).

Visualize a randomly oriented set of dipole magnets (bar magnets) buried beneath the ground with random locations and strengths. The question is this: "Can a plane of data collected near the ground surface be used to characterize the spatial

distribution, orientation, and strength of these permanent dipoles?" We propose here to investigate one buried dipole to provide a starting point, or basis, to answer this question. We further simplify the situation to two-dimensions, working in the y-z plane only for the three-dimension case using a line of data taken along the y-axis direction.

The equations for the three-dimensional vector magnetic field, when reduced to the y-z plane, are given by standard texts:

$$B_{yi}(\hat{r}_i, \alpha, \varphi, \hat{R}) = \alpha \frac{\{ \sin(\varphi) (2[y_i - Y]^2 - [z_i - Z]^2) + 3 \cos(\varphi) [y_i - Y] [z_i - Z] \}}{([y_i - Y]^2 + [z_i - Z]^2)^{\frac{5}{2}} + \varepsilon} \quad (1)$$

$$B_{zi}(\hat{r}_i, \alpha, \varphi, \hat{R}) = \alpha \frac{\{ \cos(\varphi) (2[z_i - Z]^2 - [y_i - Y]^2) + 3 \sin(\varphi) [y_i - Y] [z_i - Z] \}}{([y_i - Y]^2 + [z_i - Z]^2)^{\frac{5}{2}} + \varepsilon}$$

Here,  $\hat{r} = (y, z)$  is the field point vector which locates the data collection points,  $\hat{R} = (Y, Z)$  locates the dipole,  $\varphi$  orients the dipole in two-dimensions, and  $\alpha$  gives the strength of the dipole. The adjustable parameter  $\varepsilon$ , where  $\varepsilon \ll |\hat{r} - \hat{R}|$  is assumed, does not come from theory, but is added to "adjust" the magnetic field for visualization purposes and to prevent singularities when plotting the dipole field. This parameter is set to zero when the formulas above are used to generate data to be used for inversion calculations. Figure 1. depicts the field of a dipole plotted on *Mathematica* software. The parameter  $\varepsilon$  has been adjusted to suppress the large field component values found near the dipole center.

For one dipole, the data-model relation is:

$$\hat{D}_i = \hat{B}_i(\hat{r}_i, \alpha, \varphi, \hat{R}) + \hat{\eta}_i \quad (2)$$

Where  $\hat{B}_i = B_{yi}\hat{y} + B_{zi}\hat{z}$

and the index "i" labels the particular field point along the "line of data". The two-component vector noise:

$$\hat{\eta}_i = \eta_{yi}\hat{y} + \eta_{zi}\hat{z}$$

has components which may or may not be correlated. Typically, the noise vector is assumed to

be drawn from a probability distribution, such as an I.I.D. Gaussian or uniform. We start the formulation of the data inversion process to estimate the parameters  $\alpha$ ,  $\varphi$ , and  $R$  by stating the usual "likelihood" for the parameters based on the data-model relation and the assumption that the noise is I.I.D. Gaussian. This turns out to be a mathematically simplifying assumption, but still will usually produce good conservative results for real data.

$$P(\hat{D}_i | \hat{r}_i, \hat{R}, \alpha, \varphi, \sigma) = \left[ \frac{1}{\sqrt{2\pi\sigma^2}} \right]^2 \exp \left\{ -\frac{1}{2\sigma^2} [\eta_{yi}^2 + \eta_{zi}^2] \right\} \quad (3)$$

Here  $\eta_{yi} = D_{yi} - B_{yi}$ ,  $\eta_{zi} = D_{zi} - B_{zi}$

For the line data set  $\{\hat{D}_i\}$  the "likelihood term" becomes

$$P(\{\hat{D}_i\} | \{\hat{r}_i\}, \hat{R}, \alpha, \varphi, \sigma) = \prod_{i=1}^N P(\hat{D}_i | \hat{r}_i, \hat{R}, \alpha, \varphi, \sigma) \quad (4)$$

or

$$P(\{\hat{D}_i\} | \{\hat{r}_i\}, \hat{R}, \alpha, \varphi, \sigma) = \left[ \frac{1}{\sqrt{2\pi\sigma^2}} \right]^{2N} \exp \left\{ -\frac{1}{2\sigma^2} [(D_{yi} - B_{yi})^2 + (D_{zi} - B_{zi})^2] \right\}$$

We re-write the data-model relation in (2) in a form explicitly displaying the dipole strength parameter  $\alpha$ .

$$\vec{D}_i = \alpha \hat{b}_i + \vec{\eta}_i \quad (5)$$

Defining two new parameters  $\lambda \equiv \sum_{i=1}^N |\hat{b}_i|^2$

and  $\beta \alpha = \lambda$  allows us to put (4) in the form:

(6)

$$P(\{\vec{D}_i\} | \{\vec{r}_i\}, \vec{R}, \beta, \varphi, \sigma) = \left[ \frac{1}{\sqrt{2\pi}\sigma} \right]^{2N} \exp \left\{ -\frac{N}{2\sigma^2} \left[ \langle |\vec{D}|^2 \rangle - \frac{2}{N} \beta h + \frac{\beta^2}{N} \right] \right\}$$

where  $h \equiv \lambda^{-\frac{1}{2}} \sum_{i=1}^N \vec{D}_i \cdot \hat{b}_i$  and  $\langle |\vec{D}|^2 \rangle \equiv N^{-1} \sum_{i=1}^N |\vec{D}_i|^2$

The likelihood function (4) has been brought into the form of (6) so that we can integrate over the parameter  $\beta$  to eliminate it from further consideration. This is a theoretically correct way to reduce the number of parameters in the problem. The so-called “nuisance” parameter  $\beta$  can be estimated at the end of the problem after we have determined optimal values for the parameters  $\vec{R}$  and  $\varphi$ . The integral to be performed is:

(7)

$$P(\{\vec{D}_i\} | \{\vec{r}_i\}, \vec{R}, \varphi, \sigma) = \int_{-\infty}^{\infty} d\beta P(\{\vec{D}_i\}, \beta | \{\vec{r}_i\}, \vec{R}, \varphi, \sigma)$$

and it is to be noted that the integrand contains a probability density yet to be determined since the integration variable appears to the left of the conditional line rather than on the right as in previous expressions.

We call upon Bayes' Theorem to get an explicit form for the integrand in (7):

$$P(\{\vec{D}_i\}, \beta | \{\vec{r}_i\}, \vec{R}, \varphi, \sigma) \propto P(\{\vec{D}_i\} | \{\vec{r}_i\}, \beta, \vec{R}, \varphi, \sigma) P(\beta | \{\vec{r}_i\}, \vec{R}, \varphi, \sigma) \quad (8)$$

The above is written as a proportionality due to the omission of an unimportant normalization constant not appearing on the RHS. The density:

$$P(\beta | \{\vec{r}_i\}, \vec{R}, \varphi, \sigma)$$

is known as a prior probability density, or “prior”, for the parameter  $\beta$ . We are required to produce an explicit form for this prior if we expect to perform the integration in (7). Since data set  $\{\vec{D}_i\}$  does not appear in it, there is no reason to assume that the prior is other than uniform and, therefore, it can be ignored.

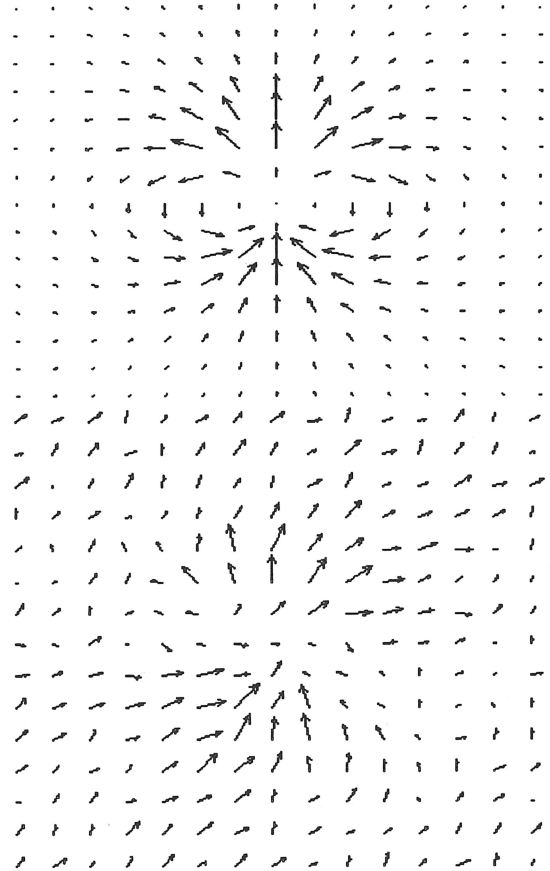


Figure 1. Dipole without noise (above) and with noise (below).

Integrating (7) gives:

(9)

$$P(\{\vec{D}_i\} | \{\vec{r}_i\}, \vec{R}, \varphi, \sigma) = \left[ \frac{1}{\sqrt{2\pi}\sigma} \right]^{2N-1} \exp \left\{ -\frac{N \langle |\vec{D}|^2 \rangle - h^2}{2\sigma^2} \right\}$$

Since the domain of  $\beta$  is all real numbers, a little thought will show that the domain of the orientation angle is  $0 \leq \varphi \leq \pi$ . Another application of Bayes' Theorem yields:

(10)

$$P(\{\vec{D}_i\}, \varphi | \{\vec{r}_i\}, \vec{R}, \sigma) = \Pi^{-1} P(\{\vec{D}_i\} | \{\vec{r}_i\}, \varphi, \vec{R}, \sigma)$$

For the present case, we assume  $\varphi$  is known and since the prior form of  $\vec{R}$  is also uniform we finally obtain:

(11)

$$P(\{\vec{D}_i\}, \varphi, \vec{R} | \{\vec{r}_i\}, \sigma) \propto P(\{\vec{D}_i\}, \varphi | \{\vec{r}_i\}, \vec{R}, \sigma)$$

It is the LHS of (11), which is to be maximized by

varying the parameters  $\varphi$  and  $\vec{R}$  to find their optimal values under the assumptions stated for the priors encountered due to Bayes' Theorem. In view of the above, if the expression:

$$h^2 = h^2(\varphi, \vec{R})$$

is maximized with respect to  $\varphi$  and  $\vec{R}$ , the same purpose is served. The question might be asked as to why it is necessary to use Bayes' Theorem to bring the parameters  $\varphi$  and  $\vec{R}$  from the right to the left side of the conditional lines in the probability density before optimizing. Indeed, a maximum likelihood or least squares process does not require this, but implicit in their standard approaches is the assumption that all priors are uniform regardless of possible prior information indicating otherwise. In the problem formulation here for a single, buried dipole we have assumed that the priors are uniform, mostly for convenience, but generally they are not. If non-uniform priors are required, then it is theoretically incorrect to proceed with a least squares or maximum likelihood formulation of an inverse problem. Incorrect results will be obtained.

We have thus achieved one of the main goals of this paper and that is to point out why a direct probability approach using Bayes' Theorem is considerably more general, due to the requirement for priors, than either least squares or maximum likelihood approaches. If cogent prior information is available before additional data is gathered, then non-uniform priors could have an effect on predicted parameter values. Least squares and maximum likelihood processes ignore priors altogether. Also such optimization processes cannot handle nuisance parameters through marginalization. The direct probability approach outlined here becomes powerful when the number of parameters to be determined becomes large, but only a few are of interest. The remainder can be, at least in theory, eliminated through integration and reducing the dimension of the problem at hand.

To complete the formulation of the single dipole inverse problem, we must obtain explicit forms for the simultaneous equations resulting from:

$$\frac{\partial h^2}{\partial \varphi} = \frac{\partial h^2}{\partial \vec{R}} = 0$$

Note that  $h$ , previously defined for (6), depends on the noisy data set  $\{\vec{D}_i\}$  and field point line  $\{\vec{r}_i\}$ , as well as  $\varphi$  and  $\vec{R}$ . Performing the derivatives results in a set of three simultaneous equations to be solved numerically for the optimal values of the location parameter:

$$\vec{R} = (Y, Z)$$

and orientation angle  $\varphi$ :

$$\sum_{i=1}^N \vec{D}_i \cdot \left\{ \lambda \frac{\partial \vec{b}_i}{\partial \varphi} - \frac{1}{2} \vec{b}_i \frac{\partial \lambda}{\partial \varphi} \right\} = 0$$

$$\sum_{i=1}^N \vec{D}_i \cdot \left\{ \lambda \frac{\partial \vec{b}_i}{\partial Y} - \frac{1}{2} \vec{b}_i \frac{\partial \lambda}{\partial Y} \right\} = 0 \quad (12)$$

$$\sum_{i=1}^N \vec{D}_i \cdot \left\{ \lambda \frac{\partial \vec{b}_i}{\partial Z} - \frac{1}{2} \vec{b}_i \frac{\partial \lambda}{\partial Z} \right\} = 0$$

It follows that when the data is noiseless, i.e.,  $\vec{D}_i \rightarrow \vec{D}_i$  (12) is identically satisfied.

A second important goal was achieved above, which was to derive a set of equations to locate and orient a dipole using noisy field data while eliminating, correctly, the need to consider dipole strength. The direct probability approach used here also has indicated the process for incorporating non-uniform priors (cogent prior information before data collection) even though for convenience we have assumed all priors to be uniform and, therefore, inconsequential. What has not been addressed is how to determine explicit prior density functions that are not uniform. The difficulty associated with producing cogent priors has been the basis for criticism of formulations of inverse problems using Bayes' Theorem. We contend that there is nothing wrong with the Bayesian approach. In fact, it is very general and powerful in its ability to correctly incorporate a variety of cogent information beyond that found in just the latest set of collected data. Further more, being able to marginalize nuisance parameters can sometimes make an intractable problem solvable.

## MATERIALS AND METHODS

This is a theoretical project in which I used probability techniques to define the parameters of a proposed model based on anomalous magnetic field data.

My method for the forward problem was to generate the data to make the picture. By this, I used the equations that we came up with and put them into *Mathematic* to create the picture of the single dipole or how ever many I decided to generate at the time and the ability to rotate the dipoles if I chose to. I then used the computer to add "noise" to the dipole. When this was done, I needed to find a way to strip off a line of data to analyze. We had a little trouble doing this because we were unsure of the exact math software to use. Finding the right math software was an adventure in itself, but when we finally decided on a program to use, it went fairly smooth.

My method for the inverse problem was to take the data generated by *Mathematic* and plug it back into Formula 12 and see if I could find the orientation, strength, and its' distribution. The process I used to do this was a function called FINDROOT. We came up with the equations to put into the computer, but

the computer solved them. Then my work was complete.

The materials I used were the three different math softwares available on campus and a computer.

## RESULTS

Formula (12) was programmed into *Mathematic* for testing and numerical solutions. Noisy data was generated using Formula (1) and then adding synthetic gaussian noise to form data lines as previously described for the two-dimensional problem. A typical signal-to-noise ratio of 1 or less was used. These data lines contained from 3 to 40 data points to be used to solve Formula (12) for dipole location. Dipole orientation was assumed known to lessen the extent of numerical calculation without sacrificing verification of the mathematical approach. It was found that beginning guesses for dipole location (required by the FINDROOT function in *Mathematic*) had to be reasonably close to the actual dipole location in order to minimize accumulated numerical error indicating that in future investigations numerical precision should be increased considerably. The formulas appeared to converge to the correct location for all tests if the initial guesses were "close enough". Since the equations are non-linear, this is not surprising. Finally, the dipole strength was eliminated from consideration as discussed and, as hoped this process did not seem to degrade location results, a main goal of this project.

## DISCUSSION

This is a project that should be taken much further. The first thing that could be done is to solve the single dipole problem in three-dimensional space. Formulating the problem in three-dimensions is more like the problem that people would face when using this in the real world. Another idea would be to have more than one dipole and make them different strengths. Someone could also try different shapes to alter the equations if they were not spherical in nature. It would be interesting to see if the equations we came up with work in more complicated situations or if problems developed, how difficult it would be to fix them if possible. Different math software could also play a part in how much farther this project could go. The programs I used were fairly powerful, but I am sure there are better ones. We discovered that the programs on campus are really slow when a big vector field is being looked at with more than a few points. Programs used in the future need to be more powerful and installed on more powerful computers.

That may improve the accuracy of the results. If I had more time, I would be interested in the three-dimensional problem along with different shapes. I think that this an interesting problem that should be looked into in the future.

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